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Hawking Radiation from the Cylindrical Symmetric Black Hole via Covariant Anomaly

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Abstract Hawking radiation from the cylindrical symmetric black hole, which is asymptotically anti-de Sitter not only in the transverse direction but also in the string or membrane direction, is discussed from the anomaly point of view. We implement the covariant anomaly cancellation method, the more refined formalism that was proposed by Banerjee and Kulkarni recently than the initial work of Robinson et al., to discuss the near-horizon gauge and gravitational anomalies. Our result shows that Hawking radiation from the cylindrical configurations with horizons also can be reproduced by the anomaly cancellation method.

Keywords Hawking radiation · Covariant anomaly · Cylindrical symmetric black hole

1 Introduction

The cylindrical symmetric black hole has obtained great interests recent years. Because it provides not only clues to the nature of the interaction between the geometry and the quantum word, such as Hawking radiation, but also an unusual space time configuration after the complete gravitational collapse has occurred classically. Due to this space time is asymptotically Anti-de-sitter in the transverse and the string directions, investigation on this black hole thus is helpful for the understanding of the Anti de Sitter/conformal field theory (CFT) correspondence, which states that one can represent the same theory either as a bulk quantum theory of gravity on dS_n or as a pure spatial conformal field theory without gravity on R^{n-1} [1]. In this paper, we investigate Hawking radiation from the cylindrical symmetric black hole, namely the black string, by the anomaly cancellation method, proposed by Robinson and Wilczek [2] and elaborated by Banerjee and Kulkarni [3] very recently. Until now, this approach has been extended to various black holes successfully [4–14]. However, all of them only involved the spherical symmetric and axial symmetric space time. To check

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the universality of this method and enforce the internal consistency of general relativity, it is necessary to extend this method to the cylindrical symmetric black hole.

As for the anomaly cancellation method, it only involves the information near the horizon. That is, anomaly takes place only at the near-horizon region of the effective field that is constructed by integrating out the classically irrelevant ingoing modes. To restore the underlying covariance at the horizon, one thus should introduce a compensating flux, which is shown to equal to that of Hawking radiation. It should be pointed out that to determine the compensating fluxes, one should employ the consistent and covariant anomalies if along the initial work of Robinson et al. Very recent, it was shown that the same work can be done by only adopting the covariant anomaly [3]. This method is believed to be conceptually clean and mathematically economical. We thus in this paper implement the refined covariant anomaly cancellation method to derive the Hawking radiation.

The remainder of this paper are outlined as follows. In the next section, we introduce some basic properties of the cylindrical symmetric black hole. Then in section three, we construct the effective field to discuss gauge and gravitational anomalies at the horizon of this hole. Section four is devoted to our concluding remarks.

2 Introduction of the Cylindrical Symmetric Black Hole

The cylindrical symmetric black hole solution of Einstein's field equation with a negative cosmological constant was found firstly by Lemos [15] about ten years ago. It was soon extended to the case with electromagnetic field [16] and rotating axes [17]. For the sake of simplicity here, we are only interested in the charged black string, which takes the form as [16]

$$ds^2 = -f dt_S^2 + f^{-1} dr^2 + r^2 d\theta^2 + \alpha^2 r^2 dz^2, \tag{1}$$

where

$$f = \alpha^2 r^2 - \frac{4M}{\alpha r} + \frac{4Q^2}{\alpha^2 r^2}, \tag{2}$$

in which, $\alpha^2 = -\frac{\Lambda}{3}$ (Λ is cosmological constant), M and Q are the mass and charge per unit length in the z direction. The electromagnetic vector potential is

$$A_u = \left(-\frac{2Q}{\alpha r}, 0, 0, 0\right). \tag{3}$$

The singularity at $r = 0$ is enclosed by the horizon $r_\pm$ when the condition $Q^2 \leq 3(M)^{4/3}/4$ holds. The inner and outer horizons are given by

$$r_\pm = \frac{1}{2}\left[\sqrt{2R} \pm \left(-2R + \frac{8M}{\alpha^3\sqrt{2R}}\right)^{\frac{1}{2}}\right], \tag{4}$$

where

$$R = \left\{\frac{M^2}{\alpha^6} + \left[\left(\frac{M^2}{\alpha^6}\right)^2 - \left(\frac{4Q^2}{3\alpha^4}\right)^3\right]^{\frac{1}{2}}\right\}^{\frac{1}{3}} + \left\{\frac{M^2}{\alpha^6} - \left[\left(\frac{M^2}{\alpha^6}\right)^2 - \left(\frac{4Q^2}{3\alpha^4}\right)^3\right]^{\frac{1}{2}}\right\}^{\frac{1}{3}}. \tag{5}$$

Fixing r at the event horizon at a constant time, the area of the black hole can be expressed as

$$\sigma_+ = \int \sqrt{-g} d\theta dz = 2\pi\alpha r_+^2. \tag{6}$$

The corresponding surface gravity at the event horizon reads off

$$\kappa_+ = \frac{1}{2}\partial_r f\Big|_{r=r_+} = \alpha^2 r_+ + \frac{2M}{\alpha r_+^2} - \frac{4Q^2}{\alpha^2 r_+^3}. \tag{7}$$

For $Q = 0$, we find the Hawking temperature of the neutral black string is $T = \frac{3\alpha}{2\pi}(\frac{M}{2})^{1/3}$, goes with $M^{1/3}$, this differs from that of the Schwarzschild black hole which is asymptotically flat. This phenomenon embodies that the difference in topology structures will impact the quantum properties of black hole configurations greatly. Though so, the laws of black hole thermodynamics however are still satisfied well [16].

3 Quantum Anomaly and Hawking Radiation

We first explore the relation between the quantum anomaly and Hawking radiation without consideration of the dynamical space time background. The most important step of this approach is the construction of the effective field by reducing the arbitrary dimensional space time to the (t, r) section of the original background with the aid of tortoise coordinate transformation and scalar field decomposition. As far as the cylindrically symmetric black hole, the effective two-dimensional metric can be written as

$$ds^2 = -f dt_S^2 + f^{-1} dr^2. \tag{8}$$

In the two-dimensional reduction, there exhibit not only gauge symmetry but also general coordinate symmetry. When the classically insignificant ingoing modes are overlooked, the effective field becomes chirial and suffers from gauge and gravitational anomalies with respect to the gauge and general coordinate symmetries respectively. The underlying theory of course is covariant; one hence should introduce fluxes to cancel the anomalies at the horizon. It must be pointed out that in this paper, we only adopt the covariant gauge and gravitational anomalies to discuss the quantum anomaly at the event horizon. This covariant anomaly cancellation method [3] is put forward very recently and is believed more refined than the initial work of Robinson and Wilczek.

We first discuss the gauge anomaly by this refined approach. The covariant form of the two-dimensional Abelian gauge anomaly [18–20] takes the form as

$$\nabla_\mu \tilde{J}^\mu = \pm \frac{e^2}{4\pi\sqrt{-g}} \varepsilon^{\mu\nu} F_{\mu\nu}, \tag{9}$$

in which, $+(-)$ corresponds to the left (right) handed field and $\varepsilon^{\mu\nu}$ represent the anti-symmetric tensor with $\varepsilon^{10} = \varepsilon^{01} = -1$. Because anomaly only takes place at the vicinity of the horizon when the classically irrelevant ingoing modes are ignored, thus we split the region outside the horizon into $[r_+, r_+ + \sigma]$ and $[r_+ + \sigma, \infty]$. The gauge current in this case in each region obeys

$$\nabla_\mu \tilde{J}^\mu_{(H)} = \frac{e^2}{4\pi} F_{rt} = \frac{e^2}{2\pi} \partial_r A_t. \tag{10}$$

$$\nabla_\mu \tilde{J}^\mu_{(0)} = 0. \tag{11}$$

Solving them, we get

$$\tilde{J}^r_{(H)} = a_+ + \frac{e^2[A_t(r) - A_t(r_+)]}{2\pi}, \tag{12}$$

$$\tilde{J}^r_{(0)} = a_0, \tag{13}$$

where a_h and a_0 are the integration constants, which respectively denote the values of the covariant gauge current at the horizon and infinity. Now that the outside of the horizon have been divided into two regions, the corresponding gauge flux hence can be expressed as the sum of these regions, namely $\tilde{J}^\mu_{(2)} = \tilde{J}^\mu_{(0)}\Theta_+(r) + \tilde{J}^\mu_{(H)}H(r)$, by the scalar step function $\Theta_+ = \Theta(r - r_+ - \sigma)$ and scalar top hat function $H = 1 - \Theta_+$. Then the Ward identity is

$$\nabla_\mu \tilde{J}^\mu = \partial_r\left(\frac{e^2}{2\pi}A_t H\right) + \left(\tilde{J}^r_{(0)} - \tilde{J}^r_{(H)} + \frac{e^2 A_t(r)}{2\pi}\right)\delta(r - r_+ - \sigma). \tag{14}$$

The first term in the right above will be canceled by the quantum effect of the classically insignificant ingoing modes. Therefore to save the underlying gauge covariance, the properties of the delta function would impose

$$a_0 = a_+ - \frac{e^2 A_t(r_+)}{2\pi}. \tag{15}$$

Physically, what we observe is infinite from the hole. Therefore, in order to get the observable gauge flux, one should find the value of a_0. After the covariant boundary condition [21] that requires the covariant flux to vanish at the horizon is enforced, it can be explicit written as

$$a_0 = -\frac{e^2 A_t(r_+)}{2\pi} = \frac{e^2 Q}{\pi \alpha r_+}. \tag{16}$$

This will be proved to be nothing but the flux of Hawking radiation, which is required to cancel the gauge anomaly at the horizon to save the underlying gauge covariance at the quantum level as shown above.

Besides gauge symmetry, the effective field also exhibit general coordinate symmetry. When the classically insignificant ingoing modes are omitted, the effective field would suffer from gravitational anomaly due to the pileup of the outgoing high frequency modes. The covariant anomaly for energy-momentum tensor is [18–20]

$$\nabla_\mu \tilde{T}^\mu_\nu = -\frac{1}{96\pi}\varepsilon_{\mu\nu}\partial^\mu R = \frac{1}{\sqrt{-g}}\partial_\mu \tilde{N}^\mu_\nu = \tilde{A}_\nu. \tag{17}$$

For the two dimensional effective field, we find

$$\tilde{N}^r_t = \frac{[2ff'' - (f')^2]}{192\pi}. \tag{18}$$

As the case of energy-momentum tensor flux, there isn't anomaly in the region $[r_+ + \sigma, \infty]$ of the effective field. But due to the existence of the electric field, the conservation equation should be modified as

$$\nabla_\mu \tilde{T}^\mu_{\nu(0)} = F_{\mu\nu}\tilde{J}^\mu_{(0)}. \tag{19}$$

On the other hand, the covariant energy-momentum tensor flux is anomaly in the near horizon region. Considering the effect of the electromagnetic field, the anomalous equation is

$$\nabla_\mu \tilde{T}^\mu_{\nu(H)} = F_{\mu\nu}\tilde{J}^\mu_{(H)} + \tilde{A}_\nu. \tag{20}$$

Inserting (12) and (13), the covariant fluxes of energy-momentum tensor in each region can be solved as

$$\tilde{T}^r_{t(0)} = e_0 + a_0 A_t, \tag{21}$$

$$\tilde{T}^r_{t(H)} = e_+ + \int_{r_+}^r dr\,\partial_r\left[a_0 A_t + \frac{e^2}{4\pi}A_t^2 + \tilde{N}^r_t\right], \tag{22}$$

where e_0 is the integration constant that corresponds to the value of the covariant flux of energy-momentum tensor at infinity. To get the explicit value of e_0, we first resort to (17), that is

$$\begin{aligned}\nabla_\mu \tilde{T}^\mu_t &= a_0\partial_r A_t(r) + \left[\partial_r\left(\frac{e^2}{4\pi}A_t^2(r) + \tilde{N}^r_t\right)H\right]\\ &\quad + \left(\tilde{T}^r_{t(0)} - \tilde{T}^\mu_{t(H)} + \frac{e^2}{4\pi}A_t^2 + \tilde{N}^r_t\right)\delta(r - r_+ - \sigma),\end{aligned} \tag{23}$$

where we have written the covariant flux of energy-momentum tensor as the contributions of two regions by the scalar step function and scalar top hat function. As for (23), the first term is the classical effect stems from the Lorenz force. The second terms will be canceled by the quantum effect of the ignored ingoing modes. Therefore, only the coefficients of the delta function vanish can restore the general coordinate covariance at the horizon, which means

$$e_0 = e_+ + \frac{e^2}{4\pi}A_t^2(r_+) - \tilde{N}^r_t(r_+). \tag{24}$$

It is still insufficient to get the value of e_0, we thus impose the boundary condition that requires the covariant flux to vanish at the horizon. The total compensating flux of energy-momentum tensor then can be written as

$$e_0 = \frac{e^2}{4\pi}A_t^2(r_+) - \tilde{N}^r_t(r_+) = \frac{e^2 Q^2}{\pi\alpha^2 r_+^2} + \frac{\pi T_+^2}{12}, \tag{25}$$

where the Hawking temperature is

$$T_+ = \frac{\kappa_+}{2\pi} = \frac{1}{2\pi}\left(\alpha^2 r_+ + \frac{2M}{\alpha r_+^2} - \frac{4Q^2}{\alpha^2 r_+^3}\right). \tag{26}$$

Our goal is to find the relation between the compensating fluxes and those of Hawking radiation. It is well known [16] that the Planckian distributions of fermions with appropriate chemical potential take the form as

$$N_{\pm e}(\omega) = \frac{1}{\exp[(\omega \pm eA_t(r_+))/T_+] + 1}. \tag{27}$$

Integrating the pure thermal spectrum, one can easily find that the compensating fluxes of gauge current and energy-momentum tensor respectively equal to those of Hawking radiation. That is to say, one can determine Hawking radiation from the cylindrical symmetric black hole by canceling the gauge and gravitational anomalies to save the underlying gauge and general covariance.

4 Concluding Remarks

Though the anomaly cancellation method has been applied to many black holes, all of them are restricted in the spherical symmetric and axial symmetric black holes. In this paper, we extended it to the cylindrical symmetric black hole and found that Hawking radiation also can be reproduced from this space time. Our work further confirms the universality of the anomaly cancellation method. Moreover, we find Hawking radiation here is only related to the horizon of the black hole but not depend on the dynamical equation of outgoing modes, it means that Hawking radiation may be only a kinematical effect and it maybe arises from the existence of underlying symmetry.

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